Time Measurements in Accelerated Frames of Reference

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I. INTRODUCTION

SINCE the time when Einstein presented a thorough analysis of the measurements of space and time in his theory of special relativity, the semantics involved in those two conceptual schemes have been perfectly defined for inertial frames. However, the effect of relative motion on length and time standards and the relationship between length and time measurements performed by different observers are by no means clearly understood when accelerated frames are considered. In this latter case, practically every writer explicitly or covertly makes one of the following two *assumptions:*

1. Acceleration in itself does not affect the rate of a standard clock or the length of a measuring rod (at least the effects of acceleration on the time and length measurements can be disregarded in a first approximation); velocity, therefore, is considered the only agent capable of altering the measurement.

2. Acceleration does not affect the vacuum speed of light.

Although it has been argued[1,2] that recent Mössbauer-effect experiments support the first of these two postulates, the assumption of either postulate is, essentially, a matter of convenience and rests on no compelling physical grounds. The difficulty in interpreting experimental checks of the time problem in accelerated frames is that they all simultaneously involve two factors: the rate of natural time measured in the accelerated system and the behavior of material systems with respect to that time, *i.e.*, the dynamics in an accelerated frame. The conventional approaches have consisted in ruling out one of these two factors by postulating a solution to the concomitant problem and then studying the other factor in connection with the phenomena of interest: Sherwin[1] assumes the emitted frequency to be representative of natural time (that is, he assumes the dynamics in the accelerated frame to be identical to that in an inertial frame) and checks it against experiment; Møller[3] assumes natural time to be proper time and proceeds to the dynamical analysis. Neither of these approaches seems fully satisfying. All that can be safely said is that the corresponding analyses of experimental data do not disprove the proper-time hypothesis (a negative result, although admittedly a valuable one).

More elaborate treatments of the accelerated frames, by resorting to the use of the theory of general relativity, do not help because the fourth space–time coordinate must be related to the physical time through the principle of equivalence or some similar procedure. In setting up such a relationship it is implicitly assumed that "proper time" is the actual time shown by a moving clock.

Several authors have advised against the loose use of the integrated proper time

$$\tau = \int (1 - v^2/c^2)^{1/2} dt ,$$

e.g., von Weyssenhoff,[4] and Fock.[5] (However, Fock[5] makes use of general relativistic proper time, which is equivalent to the integrated proper time in the case of a clock at rest in the accelerated frame.)

This paper contains the results of an investigation of the first of the pending problems; namely, the relationship between the time measurements performed in an inertial frame and in an accelerated frame. Specifically, any *a priori* assumptions, such as those stated in assumptions 1 and 2 above, are not used since such an approach would arbitrarily restrict the space–time coordinate transformation. The most general transformation relevant to a given physical situation will be derived from the generalized point of view. Such a transformation will necessarily include arbitrary functions and parameters.

[1] C. W. Sherwin, Phys. Rev. **120**, 17 (1960).
[2] H. Arzeliès, *Relativité généralisée-Gravitation* (Gauthier-Villars, Paris, 1961), Part I, pp. 121–122.
[3] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **30**, No. 10 (1955); Helv. Phys. Acta, Suppl. **4**, 54 (1956).
[4] J. von Weyssenhoff, Z. Physik **107**, 71 (1937).
[5] V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon Press Ltd., London, 1959), pp. 34, 212; see also J. Pachner, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron. Phys. **9**, 827 (1961).

These functions and parameters will be present in the physical predictions that can be derived from the transformation and will have to be determined from present or future observational data. This approach seems to constitute the only reliable physical procedure to be used in arriving at a real understanding of time and space measurements in accelerated frames. The physical interpretation of observational data will be further discussed in this context in Sec. VIII.

Only the simplest case is considered; namely, only *one spacelike dimension* and an accelerated frame in *"hyperbolic" motion* (to be defined explicitly in Sec. IV). Both the "proper time" and "conformal" approaches will be shown to be two particular cases of the general transformation and the physical meaning of these particular cases will be clarified.

The argument used in this paper is wholly kinematic and is not related to a gravitational scheme. Genuine gravitational fields are not uniform throughout space. Consequently, they cannot be eliminated at every space–time point by use of a physically meaningful coordinate transformation. In other words, the principle of equivalence is meaningful only in terms of local application in the case of actual gravitational fields. As a result of its tensor character, the Riemann tensor will vanish after the coordinate transformation if it vanished in the frame used before the transformation was applied. According to the criterion stressed by Synge,[6] such a result indicates the absence of a true gravitational field. Explicitly, this paper is limited to a study of accelerated frames in empty space–time.

II. HISTORICAL REVIEW

To this writer's knowledge, one of the two above-mentioned assumptions is introduced in all the previous attempts at a physical definition of time and length in a noninertial frame. Consequently, these studies can be divided into two lines of approach: proper time and conformal.

Proper-Time Approach

In the proper-time approach, the physical clocks are assumed to tick off the integrated proper time.

This is the most widespread assumption and it has often been related to the principle of equivalence of general relativity. In fact, this was the basis on which Einstein[7] introduced it. Born[8] applied it in his pioneering work on relativistic "rigid-body" motion wherein he considered the simple case of uniform acceleration ("hyperbolic motion"). On the basis of special assumptions, Einstein[9] derived a first-order transformation for the same case.

In a careful work that seems to have been largely overlooked by subsequent writers, Kottler[10,11] assumed, as a starting point, that "for the moving observer to consider himself at rest means that the proper spatial coordinates of every point of the comoving reference frame are constant from the point of view of that observer," and he showed, as could be expected, that his assumption was equivalent to requiring that the accelerated frame move as a "rigid" body in Born's sense. He derived an explicit transformation relating the inertial coordinates (x, y, z, t) to those in the accelerated frame (x',y',z',t') in the case of hyperbolic motion, t' being the proper time.

Uniform acceleration was again examined by Whittaker[12] through use of a limiting procedure applied to Schwarzschild's metric (a procedure that had already been used by Kottler[13]); his work was continued by Meksyn[14] and a comparable approach was used by Gottlieb.[15]

A thorough study of the proper-time transformation to a uniformly accelerated frame was performed by Møller[16] from the general relativistic point of view; this work led to the development of the same transformation as that derived in Kottler's work. Møller also showed that there can be no clock "paradox" when a space–time coordinate transformation is consistently used (a point of view that was also stressed by Fock[17]).

A space–time transformation applicable for use with any rectilinear motion was formulated by Born and Biem[18] and a detailed illuminating analysis of the implications inherent in this scheme was made by Crampin, McCrea, and McNally.[19]

Newman and Janis[20] derived a transformation that

[6] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), p. IX.
[7] A. Einstein, Jahrb. Radioakt. Elektronik **4**, 411 (1907).
[8] M. Born, Ann. Physik **30**, 1 (1909).

[9] A. Einstein, Ann. Physik **38**, 355 (1912).
[10] F. Kottler, Ann. Physik **44**, 701 (1914); **45**, 481 (1914); **50**, 955 (1916).
[11] F. Kottler, Sitzber. Wiener Akad. **125**, 899 (1916).
[12] E. T. Whittaker, Proc. Roy. Soc. (London) **A116**, 720 (1927).
[13] F. Kottler, Ann. Physik **56**, 401 (1918).
[14] D. Meksyn, Proc. Royal Soc. Edinburg **A51**, 71 (1931); Nature **160**, 834 (1947).
[15] I. Gottlieb, Nuovo Cimento **14**, 1166 (1959).
[16] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **20**, No. 19 (1943).
[17] Reference 5, p. 213.
[18] M. Born and W. Biem, Koninkl. Ned. Akad. Wetenschap. Proc. **B61**, 110 (1958).
[19] J. Crampin, W. H. McCrea, and D. McNally, Proc. Roy. Soc. (London) **A252**, 156 (1959).
[20] E. T. Newman and A. I. Janis, Phys. Rev. **116**, 1610, 1959.

is indirectly dependent on the proper-time hypothesis; indeed, the argument is based on the conventional "spatial metric tensor,"[21,22] whose validity, in turn, rests on the explicit assumption that length and time are evaluated in terms of proper length and proper time.

Singh and Pandey[23] considered an accelerated reference frame in free radial fall in a centrally symmetric gravitational field described by Schwarzschild's metric. They showed how, in principle, a space–time coordinate transformation could be developed under particular assumptions (although the actual calculations are too involved to be practicable.)

Finally, Arzeliès,[24] guided by the requirement for analogy to the Lorentz transformation, made a thoughtful guess at the space–time coordinate transformation. Actually, his guess is equivalent to the assumptions of proper-time measurement and quasirigidity of the accelerated frame previously made by Kottler and Møller. He carefully analyzed the behavior of clocks and measuring rods and furnished an extended bibliography.

Conformal Approach

In the conformal approach, the vacuum velocity of light is assumed to be unaffected by acceleration.

In the main, this assumption originated in mathematical studies of the conformal group of transformations in four-dimensional space or as a result of the influence of Milne's kinematic relativity.

Bateman[25] and Cunningham[26] established the conformal invariance of Maxwell's electromagnetic equations *in vacuo* and developed the mathematical study of general conformal transformations in four-dimensional space–time.

Kottler[11] analyzed the physical meaning of space–time conformal transformations. Included in his work are the results of former work by Ehrenfest[27] and van Os.[28]

Further developments were achieved by Schouten and Haantjes,[29,30] who stressed electromagnetic phenomena and considered uniform acceleration as a special case.

Independently, Page[31,32] defined a transformation between "equivalent" accelerated frames on a purely kinematic basis and particularized the case of uniform acceleration. His work stimulated generalizations and a group-theoretical study of conformal transformations by Bourgin[33] and Hill.[34] Gupta[35] tried to reconcile Page's transformation with the proper time approach by regauging length and time measurements. However, this procedure leads to a non-integrable transformation, which cannot be used as a space–time coordinate transformation unless exceptional care is exercised and a supplementary condition is introduced.[36]

Jones[37] suggested a simple construction for the practical use of conformal transformations.

The relationship between uniform acceleration and conformal invariance was recently studied by Vachaspati and Bali[38] in relation to the irritating problem of the radiation from a uniformly accelerated point charge.

General Approach

Kottler[11] apparently has been the first to notice that the assumptions, herein labeled as "proper-time approach" and "conformal approach," can actually be considered as definitions of two particular cases of a general problem. (However, Kottler did not study a more general case.) This remark clarifies Boardman's discovery[39] of discrepancies between Whit-

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21 C. Møller, *The Theory of Relativity* (Oxford University Press, London, 1960), Sec. 89.

22 L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), Sec. 10–4.

23 K. P. Singh and S. N. Pandey, Proc. Nat. Inst. Sci. India **A26**, 694 (1960).

24 Reference 2, pp. 297–329.

25 H. Bateman, Proc. London Math. Soc. **8**, 223, 469 (1910).

26 E. Cunningham, Proc. London Math. Soc. **8**, 77 (1910).

27 P. Ehrenfest, Koninkl. Ned. Akad. Wetenschap. Proc. **B15**, 1187 (1913).

28 C. H. van Os, Koninkl. Ned. Akad. Wetenschap. Proc. **B16**, 40 (1913).

29 J. A. Schouten and J. Haantjes, Physica **1**, 869 (1934); Koninkl. Ned. Akad. Wetenschap. Proc. **B39**, 1059 (1936).

30 J. Haantjes, Koninkl. Ned. Akad. Wetenschap. Proc. **B43**, 1288 (1940).

31 L. Page, Phys. Rev. **49**, 254 (1936).

32 L. Page and N. I. Adams, Phys. Rev. **49**, 466 (1936); *Electrodynamics* (D. Van Nostrand Company, Inc., New York, 1940), p. 128.

33 D. G. Bourgin, Phys. Rev. **50**, 864 (1936).

34 E. L. Hill, Phys. Rev. **67**, 358 (1945); **72**, 143, 236 (1947).

35 S. N. Gupta, Science **134**, 1360 (1961).

36 H. Arzeliès (reference 2) calls "non-Einsteinian" a reference frame derived from an empty inertial one by a non-integrable differential transformation. He showed (reference 2, especially Chap. 14, Secs. 2 and 3, and the final note) that (1) such a transformation is not sufficient to determine the new coordinate system and (2) the geodesic law and the general relativistic relationship between metric and matter-energy density are not valid in non-Einsteinian frames. The latter of these two remarks is obviously a consequence of the fact that the relevant coordinate transformations are more general than those used in tensor calculus; therefore, tensor covariance does not hold in such problems. Arzeliès' book seems to be the only place where a limited attempt is made at handling non-Einsteinian frames.

37 R. T. Jones (unpublished); Am. J. Phys. **28**, 109 (1960); **29**, 124 (1961).

38 Vachaspati and L. M. Bali, Nuovo Cimento **21**, 442 (1961).

39 J. Boardman, Bull. Am. Phys. Soc. **4**, 294 (1959).

taker's and Haantjes' metrics: They pertain to two different cases.

In the same context, McVittie[40] pointed out that the answer to such a problem as the celebrated "clock paradox" actually depends on the definition of "time" chosen for use in the accelerated frame. (A graphic description of an analogous situation has been presented by Dicke.[41])

III. PRELIMINARY IDENTITIES AND EQUATIONS

Basic Identities

As mentioned in Sec. I, space–time is considered here to be two-dimensional and capable of description by use of one spacelike and one timelike coordinate. Let $R(x,t)$ be an inertial frame characterized by the Minkowskian metric element

$$ds^2 = dt^2 - dx^2 \tag{3.1}$$

(where t stands for c multiplied by the time measured in that inertial frame and c is the vacuum light speed in an inertial frame). Let $R'(x',t')$ be another frame, unspecified except for the requirement that it be related to the inertial frames by a finite ("passive") coordinate transformation of positive Jacobian (see Sec. IV, postulate 2)

$$x = x(x',t'), \quad t = t(x',t') \tag{3.2}$$

(in other words, R' is "Einsteinian"[36]). The timelike coordinate t' is defined as c multiplied by the time measured in the relevant frame (this time will be called "natural time" in that frame). Similarly, x' is the expression of "natural length" in R', i.e., the value of x' assigned to each point of R' is determined by an actual length measurement in the accelerated frame. The use of these definitions permits a direct physical meaning to be assigned to the primed coordinates.

Partial derivatives of functions of several variables and ordinary derivatives of functions of a single variable will be designated throughout by subscripts (e.g., $x'_x = \partial x'/\partial x$, $x_{t't'} = \partial^2 x/\partial t'^2$, $a_{x'} = da/dx'$).

The following identities are easily verified:

$$x'_x = t_{t'}/J, \quad x'_t = -x_{t'}/J; \tag{3.3a}$$

$$t'_x = -t_{x'}/J, \quad t'_t = x_{x'}/J; \tag{3.3b}$$

$$J = x_{x'}t_{t'} - x_{t'}t_{x'}. \tag{3.3c}$$

Substituting Eqs. (3.2) into Eq. (3.1) yields the metric form in the primed frame:

$$ds^2 = g'_{11}dx'^2 + 2g'_{14}dx'dt' + g'_{44}dt'^2; \tag{3.4a}$$

$$g'_{11} = t^2_{x'} - x^2_{x'}, \quad g'_{44} = t^2_{t'} - x^2_{t'}; \tag{3.4b}$$

$$g'_{14} = t_{t'}t_{x'} - x_{t'}x_{x'}; \tag{3.4c}$$

$$-g' = g'^2_{14} - g'_{11}g'_{44} = J^2. \tag{3.4d}$$

Conformal Transformation

A coordinate transformation from one frame of reference (R) to another one (R') in metric space is said to be conformal if the components of the metric tensor in the two frames are related by

$$g_{ij} = Kg'_{ij},$$

where K is a function of the coordinates in either frame of reference.

According to this definition, the necessary and sufficient conditions for the transformation (3.2) to be conformal are

$$g'_{14} = 0, \quad g'_{11} + g'_{44} = 0, \tag{3.5}$$

that is,

$$t_{t'}t_{x'} = x_{t'}x_{x'}; \quad t^2_{t'} + t^2_{x'} = x^2_{t'} + x^2_{x'}. \tag{3.6}$$

Since x and t, in Eq. (3.2), are independent functions of x' and t', Eq. (3.6) yields

$$t_{x'} = \eta x_{t'}, \quad t_{t'} = \eta x_{x'}, \tag{3.7}$$

where η is ± 1, and can be taken to be $+1$ by appropriate orientation of one of the coordinate axes. Equation (3.7) is obviously equivalent to Jones' criterion.[37]

The velocity of light measured from R' at any event (x',t') is

$$V' = c(dx'/dt')_{ds=0}$$

and is a solution of the equation[42]

$$g'_{11}V'^2 + 2cg'_{14}V' + c^2g'_{44} = 0. \tag{3.8}$$

The two solutions are the velocities of light in both directions.

The physical meaning of the conditions (3.5) is apparent in Eq. (3.8): *A conformal transformation is a transformation such that the velocity of light, with respect to R', is equal to c in both directions at every event (x',t')*. It will be noted that conformal transfor-

[40] G. C. McVittie, Astron. J. **63**, 448 (1958).

[41] R. H. Dicke, Sci. American **205**, No. 6, 92 (1961).

[42] Equation (3.8) is different from the conventional general relativistic equation for the velocity of light (see, for instance, reference 21) because the physical meaning of the coordinates is different: in the present work, x' and t' represent the lengths and times actually measured in R', whereas in the conventional approach they have no immediate physical interpretation and the actual measurements are assumed to be expressed by proper length and proper time. The leading idea at the basis of this paper is, precisely, to waive the latter assumption.

mations, in themselves, are neither related to nor based on constant acceleration, in contradistinction to what some authors seem to believe.

Velocities

A point is defined as being at rest with respect to a reference frame when its spacelike coordinate is invariable with respect to that frame.

Let P' be a fixed point with respect to R', with abscissa x'. Its velocity $v_{P'}$ with respect to R, at some instant t', is expressed as follows:

$$w_{P'} = v_{P'}/c = (dx/dt)_{x'=\text{const}} = x_{t'}/t_{t'} . \quad (3.9)$$

Conversely, if P is a fixed point with respect to R, coincident with P' at time t', the velocity of P, as measured from R' at that instant, is, through use of Eqs. (3.3),

$$w_P' = v_P'/c = x_t'/t_t' = -x_{t'}/x_{x'} . \quad (3.10)$$

It is clear that observers in R and R' generally do not agree on the relative velocities of the (instantaneously) coinciding points P,P'. They do so only if the transformation (3.2) is such that

$$t_{t'} = x_{x'} . \quad (3.11)$$

This is the case for a conformal transformation.[43]

Acceleration

A similar analysis could be made for the acceleration. The only type of acceleration that is considered in this paper is the rest acceleration, i.e., the acceleration of a moving point P' at some instant T_0, as measured from the inertial frame $R_0(X,T)$ instantaneously at rest with respect to P' at the instant T_0. This acceleration will be computed presently.

The velocity $V_{P'}$ of P', as measured from R_0, can be obtained by use of Eq. (3.9):

$$W_{P'} = V_{P'}/c = X_{t'}/T_{t'} .$$

Since this velocity must vanish at T_0, so does $X_{t'}$.

The rest acceleration A of P' is thus expressed (as a result of setting $X_{t'} = 0$ after differentiation is completed) by

$$A/c^2 = (dW_{P'}/dT)_{dx'=0} = (\partial W_{P'}/\partial t')/(\partial T/\partial t')$$
$$= X_{t't'}/T_{t'}^2) .$$

The explicit formulas for X and T in this expression are

$$X = k(x - w_{P'}t) ; \quad T = k(t - w_{P'}x) ;$$
$$k = (1 - w_{P'}^2)^{-1/2} . \quad (3.12)$$

The velocity $w_{P'}$ must be considered a constant in the differentiations and finally replaced by its value (3.9). Use of this procedure results in

$$A/c^2 = (x_{t't'}t_{t'} - x_{t'}t_{t't'})(t_{t'}^2 - x_{t'}^2)^{-3/2} . \quad (3.13)$$

In the particular case in which the rest acceleration of every point of R' is constant, A/c^2 is a function of x' alone, say, $a(x')$. The differential equation (3.13) can then easily be integrated to

$$a[x - f_2(x')] = \{1 + a^2[t - f_1(x')]^2\}^{1/2} , \quad (3.14)$$

where f_1 and f_2 are arbitrary functions of x'. This is the general equation for the well-known hyperbolic motion. The physical significance of the arbitrary functions f_1, f_2 is the following: The value of f_1 for any given point P' is determined by the initial velocity $v_0 = cw_0$ of that point for $t = 0$, according to

$$w_0 = -af_1(1 + a^2f_1^2)^{-1/2} ; \quad (3.15)$$

f_2 is then clearly related to the initial position.

Proper Time and Length

The ratio, at T_0, of the actual infinitesimal length and time measurements dx', dt' performed in the accelerated frame, to the corresponding proper length and time measurements dX, dT are, from Eqs. (3.12), (3.9), and (3.4),

$$(dt'/dT)_{dx'=0} = (t_{t'}^2 - x_{t'}^2)^{-1/2} = g_{44}'^{-1/2} ; \quad (3.16a)$$

$$(dx'/dX)_{dT=0} = (t_{t'}^2 - x_{t'}^2)^{1/2}/(x_{x'}t_{t'} - x_{t'}t_{x'})$$
$$= g_{44}'^{1/2}/J . \quad (3.16b)$$

IV. POSTULATES

The postulates on which the argument is based can be divided into two groups. The first group is composed of fundamental general assumptions that, in effect, constitute a definition of the basic features of the present approach. The second group consists of simplifying hypotheses that are not essential to this approach but help keep the mathematical calculations within reasonable limits.

Basic Assumptions

Postulate 1. *In the domain where the transformation* (3.2) *is valid, x and t depend continuously on x', on t', and on the acceleration, and are differentiable functions of x' and t'.*

Postulate 2. *In the limiting case in which the acceleration of every point of R' vanishes, the coordinate*

[43] For example, in special relativity the reciprocity of the velocity measurements ($w_{P'} = -w'_P$) is a consequence of both the postulate of the universal constancy of the velocity of light in inertial frames and of the principle of relativity. But, it need not be maintained in a generalization where both postulates of special relativity are waived.

transformation must amount to the Lorentz transformation.

An immediate implication of the first two postulates is that *the Jacobian of the coordinate transformation* (3.2) *is positive*, i.e., [see Eq. (3.3c)],

$$J > 0 \,. \tag{4.1}$$

Postulate 3. *In order that the transformation be consistent with the usual conception of "time" and "space" coordinates, it is required that t be a monotonous increasing function of t′ and that ds^2 be positive when when dx′ vanishes,*[43a] *i.e.,*

$$t_{t'} > 0 \,, \quad g'_{44} > 0 \,. \tag{4.2}$$

Postulate 4. For an observer sitting in R' to be able to consider it as a solid frame of reference, some specification must be made as to the "rigidity" of that frame.

The wording of such a specification is by no means obvious, and indeed, several definitions of the "rigidity" of the accelerated frame have been presented in earlier papers. One such definition, general enough to include some systems worked out by earlier writers, will be chosen as the fourth postulate: *The velocity of light in each direction, in each point of R', is independent of time from the point of view of R'.* A consequence of this assumption is that the frequency of an electromagnetic radiation is propagated without change inside the (empty) frame R' when such frequency is measured in terms of units of R'. The above definition of rigidity is the natural extension of Born's special-relativistic criterion, akin to that proposed by Synge[44] although somewhat more stringent.

It can be seen in Eq. (3.8) that the mathematical expression of postulate 4 is

$$g'_{ij} = f_{ij}(x')f(x',t') \,, \quad (ij = 11,14,44) \,. \tag{4.3a}$$

These three conditions entail the additional equation

$$J = \varphi(x')f(x',t') \,, \tag{4.3b}$$

which may take the place of any one of Eqs. (4.3a). It will be convenient to use this latter equation instead of the first of (4.3a).

It will be seen in Sec. VI that the proper-time approach corresponds to a particular case of postulate 4 with $f(x't')$ reduced to a constant. In that case, the "rigidity" criterion developed by Newman and Janis[20] and by Arzeliès,[45] namely, that the conventional "spatial metric tensor" is time independent, is satisfied by postulate 4. The same criterion is equivalent to that of Rosen,[46] as was proved by Newman and Janis.[20]

Of the four postulates in this group, the first three can practically be inferred from the physical meaning of the transformation, and these three have been accepted by every earlier writer. The actual generalization is to be found in the fourth postulate. The latter may be considered controversial. In this writer's opinion, its virtue consists in the fact that its use seems to minimize the *a priori* restriction imposed upon the transformation, but, of course, it might have to be replaced by an even more general postulate, or perhaps by a different one, if it happened to be disproved in an experiment.

Simplifying Hypotheses

It turns out that the most general solution of the problem defined by the four postulates above, without any restrictive assumption at all, is practically unmanageable. Instead of placing arbitrary conditions on the relations between measurements, which are the *result* in this problem, it has been found convenient to introduce the necessary simplifications in the *starting point* where their physical bearing can be readily assessed.

The first obvious step is to reduce the number of dimensions:

Postulate 5. *Only one spacelike dimension is considered.*

A second simplification consists of assuming that the law of motion, as seen from an inertial frame, of each point that is at rest with respect to the accelerated frame is known and is described by means of an explicit equation in space–time. This assumption obviously involves no restriction in any given physical situation. The limitation consists only in the inconvenience that the solution must be worked out anew if the kinematic conditions are changed. The same idea can be readily generalized to the four-dimensional space–time.

The particular law of motion to be preferred in a first study is suggested by the following argument. Special relativity is the theory of the effects, on kinematic and dynamic measurements, of relative velocity with respect to an inertial frame. The method used in special relativity to ensure that rigorous results are obtained consists in considering

[43a] Actually, the latter condition is equivalent to the statement that the velocity of the accelerated observer with respect to R is less than c. Since this statement is necessarily true when postulates 6 and 7 are accepted, the condition $g'_{44} > 0$ may be omitted in the present problem. Also, the condition $t_{t'} > 0$ is implied by postulates 1 and 2 if $t_{t'}$ is continuous.

[44] Reference 6, p. 115.

[45] Reference 2, p. 71.

[46] N. Rosen, Phys. Rev. **71**, 54 (1947).

a nonaccelerated moving frame. Any possible effect of acceleration is thus eliminated without any approximating assumptions having to be made. The results may then be tentatively applied as an approximation when small accelerations are involved; such a procedure leads, precisely, to the "proper-time approach."

The scope of the proper-time approach is thus clearly brought to light; it is, so far, an *approximate treatment*, valid for small enough accelerations, and it must be considered as such until the physical effects of acceleration are fully understood. The phrase "small-enough accelerations" is quite vague. Its operational meaning is: that range of accelerations in which experiments are in acceptable agreement with the theoretical predictions. This range may possibly cover all accelerations of interest (indeed no presently available experiment seems to show that the boundary is exceeded), or it may even cover any acceleration whatsoever (if clock rates and measuring-rod lengths are actually insensitive to acceleration); but this question must be decided by unprejudiced investigation.

The obvious way to extend the method of special relativity to acceleration effects is to eliminate any spurious influence from a variation of acceleration by using only a constant acceleration. The results may then be tentatively applied as an approximation in the case of slowly varying acceleration and this approximation may be expected to be better than that obtained by use of the proper-time formalism.

Acceleration is not a well-defined quantity in this problem; the acceleration of P' has a different value according to whether it is measured from R, R', or R_0. There does not appear to be any physical reason to prefer any one of these three points of view. Indeed, whatever the choice, a constant acceleration is by no means a convenient case for experimentation, but there are advantages in considering a constant rest acceleration: (1) it is expressed by a finite equation (3.14) for the unknown functions (3.2), and (2) since the case of constant rest acceleration has been considered by several authors (Sec. II), the comparison of the different known particular cases with the general formula to be derived is straightforward. Therefore, the following assumption is made.

Postulate 6. *Every point at rest in R' undergoes hyperbolic motion with respect to R*, or, in other words, the functions (3.2) satisfy Eq. (3.14).

It will be noted that no assumption is made as to the values of the accelerations of the different points in R'; the function $a(x')$ is left completely unrestricted. Any assumption regarding a would involve a physical hypothesis concerning the behavior of R', as seen from R, and the use of such a hypothesis would bias the argument.

The same can be said of $f_2(x')$. This function describes the relationship between the distributions of length measurements in R and R' at some (arbitrary) initial moment. It must be left unrestricted.

As for $f_1(x')$, it was seen [Eq. (3.15)] that this function is related to the distribution of velocities among the points of R' at an initial time which is again arbitrary. In principle, it may also be any function, but the calculations will be appreciably simplified by further specifying the physical problem as follows.

Postulate 7. It will be assumed that *there exists one instant of time at which all the points of R' are simultaneously at rest with respect to R.*

That such an assumption is physically reasonable appears from the fact that it represents the actual situation when two (inertial) reference frames are at rest with respect to each other and one of them is suddenly accelerated. Such a situation would be expected to occur in any experiment on accelerated systems where two identically constructed clocks are found at rest in an inertial frame at the beginning or end of the process. If the particular instant described in postulate 7 is taken as the time origin, the mathematical expression for the postulate is [from Eq. (3.15)]

$$f_1(x') \equiv 0\,. \tag{4.4}$$

This seventh postulate could be waived at the price of facing more involved computations, but the argument would not be essentially different. A straightforward partial generalization is the case of a nonzero velocity which is the same for all the points in R'. The relevant transformation is then the product of a Lorentz transformation and the transformation to be derived hereafter.

In the sixth and seventh postulates there is no implication that the length and time measurements are identical in both frames at the initial time, for the assumption is not made (as it is in the conventional papers and textbooks) that those measurements are acceleration independent. Thus the simplification introduced by use of these assumptions is less stringent than the simplifications resulting from the assumptions usually made by earlier writers who studied the one-dimensional hyperbolic motion. Specifically, the generalization used in this paper consists of allowing f_2 to be an undetermined function of x'.

V. THE GENERAL TRANSFORMATION

The most general transformation that satisfies Eqs. (3.14) and (4.4) (i.e., Postulates 5–7) is obviously

$$x = f_2 + a^{-1}\cosh F\ ;\quad t = a^{-1}\sinh F\ , \tag{5.1}$$

where F is a completely undetermined function $F(x',t')$ of x' and t'.

These functions, substituted into Eqs. (3.4), lead to the metric tensor

$$g'_{11} = (F_{x'}/a)^2 - (a_{x'}/a^2)^2 + (f_{2,x'}/a^2) \times (a_{x'}\cosh F - aF_{x'}\sinh F) - f^2_{2,x'}\ , \tag{5.2a}$$

$$g'_{44} = (F_{t'}/a)^2\ , \tag{5.2b}$$

$$g'_{14} = F_{x'}F_{t'}/a^2 - (1/a)f_{2,x'}F_{t'}\sinh F\ , \tag{5.2c}$$

$$J = -a_{x'}F_{t'}/a^3 + (1/a)f_{2,x'}F_{t'}\cosh F\ , \tag{5.2d}$$

where $a_{x'}$ and $f_{2,x'}$ are the derivatives of a and f_2, and $F_{x'}$, $F_{t'}$ are the partial derivatives of F.

The second of the Eqs. (4.2) is obviously satisfied; from the first one, it can be inferred that

$$aF_{t'} > 0\ . \tag{5.3}$$

Consequently, condition (4.1) indicates that

$$-a_{x'}/a^2 + f_{2\,x'}\cosh F > 0\ . \tag{5.4}$$

A transformation described by Eqs. (5.1), (5.3), and (5.4) satisfies postulates 1, 3, and 5–7. Postulate 2 contains a prescription for boundary values, when $a(x') = 0$, which will have to be worked out for every particular case. The transformation will now be further restricted through the use of postulate 4.

This restriction can be effected by expressing that the ratios g'_{14}/g'_{44} and J/g'_{44} are functions of x' alone, i.e.,

$$F_{x'}/F_{t'} - (a/F_{t'})f_{2,x'}\sinh F = f_3(x')\ , \tag{5.5}$$

$$-a_{x'}/aF_{t'} + (a/F_{t'})f_{2,x'}\cosh F = f_4(x')\ . \tag{5.6}$$

[It will be noted that f_4 is positive on account of Eqs. (4.1) and (4.2).]

The system of equations (5.5) and (5.6) is readily solved in the particular case in which $f_{2,x'} = 0$. This solution is described in Sec. VI. *In the rest of this section, $f_{2,x'}$ is assumed to be nonzero.*

In this case the derivation is rather lengthy and is deferred to Appendix 1. The result is the following transformation:

$$x = \epsilon\lambda^{-1}[\mu D^{-1}\alpha(x') + a^{-1}(0)]\ , \tag{5.7a}$$

$$t = -\mu^{1/2}D^{-1}\sinh[\mu^{1/2}a(0)(t'+f_5)]\ , \tag{5.7b}$$

$$D = a(0)\{\lambda\cosh[\mu^{1/2}a(0)(t'+f_5)] - [\lambda^2 + \mu\alpha^2(x')]^{1/2}\}\ , \tag{5.7c}$$

$$\alpha(x') \equiv a(x')/a(0)\ ,\quad \epsilon = \pm 1\ ,\quad \mu = 1 - 2\lambda\ , \tag{5.7d}$$

where λ is a parameter bound by the conditions

$$0 < \lambda \leqslant \tfrac{1}{2}\ ; \tag{5.8}$$

f_5 designates a function of x', arbitrary except for the restriction

$$f_5(0) = 0\ . \tag{5.9}$$

A restriction is also placed on the acceleration function $\alpha(x')$: when $a(0)$ tends to zero, $\alpha(x')$ must behave as

$$1 + \epsilon(1-\lambda)a(0)x'\ . \tag{5.10}$$

The corresponding metric form is

$$ds^2 = \mu^2 a^2(0)D^{-2}\{dt'^2 + 2f_{5,x'}dx'dt' + [f^2_{5,x'} - a^{-2}(0)\alpha^2_{x'}(\lambda^2+\mu\alpha^2)^{-1}]dx'^2\}\ . \tag{5.11}$$

In the limiting case in which λ tends to 1/2, the transformation becomes

$$x = 2\epsilon a^{-1}(0)[\alpha D'^{-1} + 1]\ , \tag{5.12a}$$

$$t = -D'^{-1}(t' + f_5)\ , \tag{5.12b}$$

$$D' = \tfrac{1}{4}a^2(0)(t'+f_5)^2 - \alpha^2\ . \tag{5.12c}$$

This transformation could also be computed by direct integration of Eq. (A10) if $k = 0$. The corresponding metric form is

$$ds^2 = D'^{-2}\{dt'^2 + 2f_{5,x'}dx'dt' + [f^2_{5,x'} - 4a^{-2}(0)\alpha^2_{x'}]d^2_{x'}\}\ . \tag{5.13}$$

Use will also be made later of the other limiting case in Eqs. (5.7), in which λ tends to 0:

$$x = -\epsilon\{a^{-1}\cosh[a(0)(t'+f_5)] - a^{-1}(0)\}\ , \tag{5.14a}$$

$$t = a^{-1}\sinh[a(0)(t'+f_5)]\ . \tag{5.14b}$$

VI. THE PROPER-TIME APPROACH AS A SPECIAL CASE

The derivation in Sec. V is based on the assumption that $f_{2,x'}$ does not vanish. The case of a vanishing $f_{2,x'}$ will now be investigated. If $f_{2,x'}$ vanishes, Eqs. (5.6) and (5.2b) show that $F_{t'}$ and g'_{44}, and therefore g'_{ij} and J, are functions of x' alone. Consequently, Eqs. (3.16) indicate that at any point of the accelerated frame R' both the ratio of local natural time to local proper time and the ratio of local natural length to local proper length are independent of time. In other words, at every point in R' natural time is essentially equivalent to proper time; this is the proper-time approach.

Conversely, in the proper-time approach g'_{44}, and therefore $F_{t'}$, must be functions of x' alone. Since $\cosh F$ cannot be independent of t' [see Eq. (5.1)], Eq. (5.6) entails $f_{2,x'} = 0$. Thus, $f_{2,x'} = 0$ *is the necessary and sufficient condition for the proper-time solution.*

Since $F_{t'}$ is independent of t', so is $F_{x'}$ as a consequence of Eq. (5.5), and F has the form

$$F = b[t' + \psi(x')] , \tag{6.1}$$

where b is a constant and $\psi(x')$ is an undetermined function of x'. Introduction of (6.1) and of the initial conditions (A15) into the transformation (5.1) yields

$$x = a^{-1} \cosh [b(t' + \psi)] - a^{-1}(0) , \tag{6.2a}$$

$$t = a^{-1} \sinh [b(t' + \psi)] . \tag{6.2b}$$

Application of the condition that the transformation must take on the limiting form $x = x'$, $t = t'$ in the limit when $a(0)$ tends to 0, yields

$$b = a(0) , \tag{6.3}$$

and shows that, when $a(0)$ tends to zero, a must behave as

$$a(0)[1 - a(0)x'] . \tag{6.4}$$

It is clear that Eqs. (6.2)–(6.4) are identical with Eqs. (5.14) and (5.10) in which $\epsilon = -1$, $\lambda = 0$. Therefore, the proper time solution is included as a limiting case in the general transformation (5.7) through (5.10). The latter is the most general transformation consistent with the postulates chosen in Sec. IV and the full range of λ is now described in the following equation, which takes the place of Eq. (5.8):

$$0 \leqslant \lambda \leqslant \tfrac{1}{2} \tag{6.5}$$

One physical peculiarity of the proper-time transformation (6.2) is that for a given x' the relation between t and t' is singularity free and the range $(-\infty, +\infty)$ of t corresponds to the range $(-\infty, +\infty)$ of t'. For any nonzero value of λ, there are two singularities $t' = f_5 \pm t'_1$ in the t function, and the range $(-\infty, +\infty)$ of t is covered by the range $(f_5 - t'_1, f_5 + t'_1)$ of t'. This is a particular aspect of the more general fact that the larger λ, the larger the time dilation in the accelerated frame; that is, the time dilation is least in the proper-time solution and greatest[46a] in the case of the rational transformation (5.12). This property can easily be visualized by roughly plotting t against t' for a constant x'.

As an example, take the case of strongest time dilation [Eq. (5.12)] and assume the acceleration at $x' = 0$ to be 1 g. The value of t'_1 at the origin of R' is then $2/g$, which means a (natural) time of the order of two years. In the same case, for example, the finite value of the "age of the universe" could be accounted for if our astronomical frame of reference had an average uniform rectilinear acceleration of the order of $10^{-10}g$ with respect to the true inertial frames.

It will be noticed that the relation stated immediately after Eq. (6.5) is not reciprocal and that for a constant x (fixed point in R instead of R') there is a singularity (in the proper-time solution) in the expression of t' as a function of t. The transformation (6.2), (6.3) can easily be inverted, and the singularity is found to take place at $t = \pm [x + a^{-1}(0)]$.

Another physical characteristic of the proper-time solution (which is easily understood from the definition that was given above for this particular case) is the following:

The requirement for "rigidity" of R' in Born's sense (i.e., that there exists at any time an inertial frame with respect to which every point of the accelerated frame is instantaneously at rest) is equivalent[47] to requiring that the paths (in R) of the different points of R' be concentric homothetic hyperbolas. The necessary and sufficient condition for such a situation is $f_2 =$ const, as can be derived from Eqs. (3.14) and (4.4). *What is called here the "proper-time solution" is thus determined by adding the requirement of Born's "rigidity" to the set of postulates of Sec. IV.*

VII. PARTICULAR CASES

The general transformation (5.7) through (5.10) depends on a parameter λ, ranging from 0 to 1/2, and on two arbitrary functions $a(x')$ and $f_5(x')$, submitted to limiting restrictions (5.9) and (5.10). The first function $a(x')$ represents the distribution of proper accelerations among the points of R', while $f_5(x')$ describes the synchronization of the natural clocks in R'. Let a few particular cases be considered.

1. Vanishing of f_5

The specification

$$f_5 \equiv 0 \tag{7.1}$$

is a necessary and sufficient condition for each of the following properties: (a) Speed of light in R' is direction independent at every point and (b) the

[46a] G. J. Whitrow [*The National Philosophy of Time*, Thomas Nelson and Sons, New York (1961), p. 219, footnote 1] has remarked that the time dilation associated with Page's transformation is greater than the dilation corresponding to the proper-time transformation.

[47] J. E. Romain (unpublished).

synchronization of distant clocks at rest is acceleration independent (i.e., $t = 0$ and $t' = 0$ imply each other, whatever x'). Condition (b) has generally been assumed by earlier writers and condition (a) was then derived as a consequence. Actually, it can be seen directly that these two conditions are equivalent.

2. Conformal Transformations

The necessary and sufficient conditions (3.5) for a conformal transformation become, in the case of the general metric form (5.11),

$$f_5 \equiv 0\ ; \quad \alpha_{x'}^2 = a^2(0)(\lambda^2 + \mu\alpha^2) \qquad (7.2)$$

The latter equation can be integrated to

$$\alpha(1-\lambda) - (\lambda^2 + \mu\alpha^2)^{1/2} = \lambda^2\mu^{-1/2}\sinh[\zeta\mu^{1/2}a(0)x'],\quad \zeta = \pm 1\,. \qquad (7.3)$$

The conformal transformations in this problem are, thus, elements of a one-parameter set, with parameter λ. In the limit when λ is 1/2, Eq. (7.3) becomes

$$\alpha = 1 + \tfrac{1}{2}\zeta a(0)x'\,,$$

and the transformation (5.12) becomes identical to Page's transformation.[31,47a] Thus, *Page's transformation is the conformal transformation pertaining to the limiting case* $\lambda = 1/2$.

In the other limiting case, when λ vanishes, Eq. (7.3) must be replaced by

$$\alpha = \exp[\pm a(0)x']\,.$$

No conformal transformation seems to have been used to date with a value of λ different from 1/2.

Starting from the notion of observer equivalence in the sense of Milne, Bourgin[33] suggested to generalize Page's transformation to a class of conformal transformations characterized by the form

$$ds^2 = (dt'^2 - dx'^2) \times G(t-x)G(t+x)/G(t'-x')G(t'+x')\,, \qquad (7.4)$$

where G is an arbitrary function. The present conformal transformation [(5.7) through (5.10) and (7.2)] cannot be put into that form unless λ is 1/2 (see Appendix 4). Thus, *the only Bourgin transformation consistent with the present postulates is Page's.*

[47a] It has been seen in Sec. IV (postulate 7) that the time t' defined by the present postulates is not homogeneous, and that the instant defined as time origin after postulate 7 is particularized. This feature seems related to the "discrepancy" discovered in Page's transformation by E. A. Milne and G. J. Whitrow [Z. Astrophys. **15**, 342 (1938)].

3. Proper-Time Transformation

The proper-time solution ($f_2 =$ const, $\lambda = 0$) is expressed by Eqs. (6.2) and (6.3).

The various earlier particular solutions of this type were generally based on the assumption $f_5 \equiv 0$. Therefore, the only difference between them is the form of the acceleration function a and they can be reduced to each other (in the one-dimensional case) by a transformation effected on x' alone.

The transformation studied by Kottler,[11,48] *Møller,*[1] *Crampin et al.,*[19] *and Arzeliès*[24] is characterized by

$$f_5 \equiv 0\,, \quad g'_{11} = -1\,.$$

The latter condition was explicitly postulated by Møller; it can obviously be fulfilled only in the proper-time case ($\lambda = 0$). This condition is usually related to the assumption of a Euclidian three-dimensional spatial geometry when the other two spatial dimensions are assumed to be unaltered. It entails [see Eq. (5.11)]

$$\alpha = [1 - \epsilon a(0)x']^{-1}\,.$$

The same transformation has also been derived by Newman and Janis[20] as a particular case [their Eq. (3.20) with $n = a(0)\bar{t}$]; but the similarity with these authors' work does not go any further, for their approach is basically different from the present one (see Sec. IV).

The condition $g'_{11} = -1$ considered above has the following geometrical meaning: It is equivalent, in the present one-dimensional problem, to the requirement that the infinitesimal natural length dx' be identical to the metric element $[(-g'_{11})^{1/2}\,dx']$ induced on the x' axis by the space–time metric.[49]

The transformation derived by Kottler,[50] *Whittaker,*[12] *Meksyn,*[14] *and I. Gottlieb*[15] is characterized by

$$f_5 \equiv 0\,, \quad g = ||g_{ij}|| = -1\,.$$

The condition $g = -1$, which was explicitly postulated by Gottlieb, is equivalent to the requirement that the Jacobian of the transformation be unity. It entails $\lambda = 0$ and

$$a = a(0)[1 - 2\epsilon a(0)x']^{-1/2}\,.$$

VIII. CONCLUSION

Although the results derived for the simple case considered in this paper could hardly be checked by an experiment and do not seem to have any im-

[48] Reference 13, Sec. 34.
[49] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1960), p. 45.
[50] Reference 13, Sec. 35.

mediate practical use, the investigation of such a case is helpful in making clear that the definition of "natural time" in an accelerated frame is by no means a simple and obvious matter. Full realization of this fact is the first step towards an improved interpretation of physical evidence.

The presently available experiments are based on the emission, transmission and reception of radiation of some frequency. Tonnelat[51] has shown that the very definition of frequency must be worded carefully. She discerns three kinds of "frequencies" for a given phenomenon: the frequency $(\nu_i)_j$ of a vibration from a source S_i as measured by an observer S_j; the "relative proper frequency" $(\nu_i)_i$ measured in the source itself in terms of its own natural time; and the "absolute proper frequency" ν_{00}, which is the number of emitted cycles per unit of the source's proper time. [In this paper's terminology $(\nu_i)_i$ would better be called "natural frequency" and ν_{00} "proper frequency."] As was pointed out by Tonnelat, the actual comparisons of frequencies (shift measurements) that are available to date have consisted in comparing either $(\nu_i)_j$ and $(\nu_j)_j$, or $(\nu_i)_i$ and $(\nu_j)_j$.

The correct interpretation of such shift-measurement experiments as those reviewed by Sherwin[1] implies consideration of three items: (1) the relevant definition of the source's and observer's natural times; (2) the behavior, with respect to these times, of the particular timing devices used; and (3) the transmission of frequency.

The argument in this paper was an introduction to the study of the first of these problems. The second problem has been painstakingly studied by Møller[3] on the basis of the assumption that natural time is identical to proper time. He derived the conditions under which a material clock is an "ideal standard clock" (i.e., ticks off proper time) to a given approximation. Those conditions involve mass of the moving system, amplitude of vibration, frequency, speed, and acceleration of the clock. They are satisfied by currently available "atomic clocks" within present experimental accuracy but Møller's analysis makes it quite clear that his approach is only an approximation based on simplifying assumptions. Fokker[52] proposed a model of a light-ray clock that is unaffected by acceleration if it undergoes a hyperbolic motion. However, his result is hardly surprising in the light of the analysis developed above: A one-dimensional section of Fokker's device complies with the postulates in Sec. IV and, moreover, Born's rigidity is assumed. It has been seen (Sec. VI) that these are the necessary and sufficient conditions for the proper-time solution. Thus, the beats of Fokker's "clock" in hyperbolic motion are isochronous because they consist in timing, in proper-time units, a light-ray in a Born-rigid system; such a procedure implies *assuming* that the acceleration is irrelevant.[53]

The third question mentioned (transmission of frequency) does not seem to raise any special difficulty since it can be handled in a single frame; for instance, in this paper's analysis frequency is propagated unaltered both in the inertial frame and in the accelerated frame (because of postulate 4).

The final solution of the problem of correctly interpreting experimental evidence is not yet within easy reach. It appears to involve the study of the following:

1. The present analysis should be generalized to four-dimensional space–time and to more realistic types of acceleration better suited to experimental check (e.g., harmonic motion, uniform circular motion).

2. Dynamics must be developed in the relevant accelerated frames without any physically unwarranted assumptions. A possible approach, or even a bias to bypass this step, might be the study of the system under investigation in an inertial frame, followed by transformation of the results to the accelerated frame. Such a procedure would actually be similar to Møller's analysis.[3]

3. Experimental checks must be devised and performed. These checks may include direct time measurements and frequency-shift measurements such as those discussed above. Examples of the former might be experimental outcomes of the celebrated twin problem when such results become available, or evaluations of spans of time comparable to the age of the universe (see the end of Sec. VI). Although there is not sufficient ground in Sec. VI to substantiate a claim that the finite age attributed to the universe could be a consequence of an acceleration of our cosmological environment with respect to the inertial frames, the possibility of some related effect must not be discarded *a priori* and might well help solve some discrepancies. Another possible check might be related to the "gravitational red shift" in an accelerated frame. In the conventional approach[16]

[51] M. A. Tonnelat: Ann. Inst. Henri Poincaré **17**, 59 (1961).

[52] A. D. Fokker, Koninkl. Ned. Akad. Wetenschap. Proc. **B59**, 451 (1956).

[53] However, in view of postulate 4, a photon that bounces back and forth *in vacuo* over a fixed limited segment of the x' axis does slice off equal intervals of time between successive passages at the same point in the same direction. In other words, a one-dimensional Fokker clock remains a "natural light clock" if both the assumption of Born's rigidity and the assumption of identity between natural time and proper time are waived.

in which measurements are performed with "ideal standard clocks" that show proper time, a light ray is expected to experience a red shift, just as it would in a gravitational field while traveling in the direction of acceleration. On the other hand, if postulate 4 is acceptable, no such effect can appear when the frequencies are measured in terms of natural time.

The extension of the method to four-dimensional space–time and to other types of motion is being investigated and will be the subject of a subsequent paper.

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APPENDIX 1. DERIVATION OF EQS. (5.7)–(5.10)

If Eq. (5.6) is multiplied by $F_{t'}$ [which cannot be zero if the transformation (5.1) is to be meaningful at all], and then differentiated with respect to t', the following equation is obtained:

$$af_{2,x'}F_{t'} \sinh F = f_4 F_{t't'} . \qquad \text{(A1)}$$

The transformation equations (5.1) can then be written, by use of Eqs. (5.6) and (A1), in the more convenient form

$$x = f_2 + (f_4 F_{t'} + a_{x'}/a)/a^2 f_{2,x'} , \qquad \text{(A2)}$$

$$t = f_4 F_{t't'}/a^2 f_{2,x'} F_{t'} . \qquad \text{(A3)}$$

It will be noticed that the transformation is determined through knowledge of the function $F_{t'}(x',t')$ and that it therefore will not be necessary to compute F itself.

The key function $F_{t'}$ will now be derived from Eqs. (5.5), (5.6), and (A1); the latter is a consequence of Eq. (5.6).

Eliminating the undifferentiated F between Eqs. (5.6) and (A1) yields

$$u_{t'}^2 = \{[f_4 + (a_{x'}/a^2)u]^2 - f_{2,x'}^2 u^2\} a^2/f_4^2 ; \qquad \text{(A4)}$$

$$u = a/F_{t'} . \qquad \text{(A5)}$$

Multiplying Eq. (5.5) by $F_{t'}$, differentiating with respect to t', and substituting cosh F from Eq. (5.6) results in

$$u_{x'} = f_3 u_{t'} - af_4 . \qquad \text{(A6)}$$

Equations (A4) and (A6) must be compatible in the unknown function u; therefore (see Appendix 2),

$$f_4 = \epsilon a_{x'}(h + ka^2)^{-1/2} , \qquad \text{(A7)}$$

$$f_{2,x'} = \epsilon'(a_{x'}/a^2)h^{1/2}(h + ka^2)^{-1/2} , \qquad \text{(A8)}$$

where ϵ is the sign of $a_{x'}$ (this sign is invariant throughout the spacelike axis), $\epsilon' = \pm 1$, and h and k are integration constants.

Since f_4 must be real, whatever the value of a, and $f_{2,x'}$ must not vanish, the constants h and k must satisfy the conditions

$$h > 0 , \quad k \geqslant 0 . \qquad \text{(A9)}$$

When the expressions (A7) and (A8) of f_4, $f_{2,x'}$ are used in Eq. (A4), the latter becomes

$$u_{t'}^2 = ku^2 + 2\epsilon u(h + ka^2)^{1/2} + a^2 . \qquad \text{(A10)}$$

This equation is immediately integrable in the particular case in which $k = 0$, but as the corresponding solution can be obtained as a limiting case from the more general case $k \neq 0$ [see Eqs. (5.12)], it is sufficient to consider only the latter.

Equation (A10) can easily be integrated to

$$u = k^{-1}\{(\epsilon'' h^{1/2}) \cosh [k^{1/2}(t' + f_5)] - \epsilon(h + ka^2)^{1/2}\} , \qquad \text{(A11)}$$

where $f_5(x')$ is an arbitrary function of x', and $\epsilon'' = \pm 1$.

In order to maintain consistency, the solution should remain meaningful when k tends to 0; the necessary condition is

$$\epsilon'' = \epsilon .$$

The solution (A11) satisfies Eq. (A6) with $f_{5,x'} = f_3$. Equation (A8) can be integrated to

$$f_2 = -(\epsilon'/a)h^{-1/2}(h + ka^2)^{1/2} + q ; \qquad \text{(A12)}$$

q is an integration constant.

The coordinate transformation (A2) and (A3) thus becomes

$$x = \epsilon' akh^{-1/2}/E + q , \qquad \text{(A13a)}$$

$$t = -(\epsilon\epsilon' k^{1/2}/E) \sinh [k^{1/2}(t' + f_5)] , \qquad \text{(A13b)}$$

$$E = h^{1/2} \cosh [k^{1/2}(t' + f_5)] - (h + ka^2)^{1/2} . \qquad \text{(A13c)}$$

This transformation satisfies Eq. (3.14) simplified by (4.4), and Eq. (4.1); it satisfies Eqs. (4.2) if

$$\epsilon = \epsilon' . \qquad \text{(A14)}$$

The origin of the x',t' coordinates may be chosen, for convenience, so that

$$x = 0 , \quad t = 0 \text{ corresponds to } x' = 0 , \quad t' = 0 . \qquad \text{(A15)}$$

Such a choice results in

$$f_5(0) = 0 , \qquad \text{(A16)}$$

$$q = -\epsilon' a(0)kh^{-1/2}/\{h^{1/2} - [h + ka^2(0)]^{1/2}\} . \qquad \text{(A17)}$$

It is now apparent that the determination of the sign of ϵ (or ϵ') is simply related to the relative orientation of the x and x' axes and is, therefore, physically irrelevant.

The integration constants h and k can finally be determined by use of postulate 2. For that purpose, in view of the constraints of Eq. (A15) and postulate 7, it must be required that Eqs. (A13) take on the limiting form $x = x'$, $t = t'$ when the acceleration vanishes. This requirement leads (see Appendix 3) to

$$h^{1/2} = \lambda a^2(0)\ , \quad k = \mu a^2(0)\ , \qquad \text{(A18)}$$

$$2\lambda + \mu = 1\ , \qquad \text{(A19)}$$

whereas $a(x')$ is subject to a limiting condition (A29).

On account of the conditions (A9) and (A19), the following limitations are imposed on λ and μ:

$$0 \leqslant \mu < 1\ , \quad 0 < \lambda \leqslant \tfrac{1}{2}\ . \qquad \text{(A20)}$$

When all the preceding results are gathered, the desired transformation is finally expressed as stated in Eqs. (5.7) through (5.10).

APPENDIX 2. DERIVATION OF EQS. (A7) AND (A8)

The requirement that $(u_{t'})_{x'} = (u_{x'})_{t'}$, applied to Eqs. (A4) and (A6), yields a condition of the form

$$B_1(x')u^2 + B_2(x')u \equiv 0\ . \qquad \text{(A21)}$$

Now u must actually depend on t'. Otherwise, $F_{t'}$ would be independent of t' and Eq. (A1) would entail either $f_{2,x'} = 0$ or $F_{t'} = 0$; the former has been excluded as a particular case treated separately in Sec. VI, and the latter is obviously inadmissible in Eqs. (5.1). Thus, Eq. (A21) can be satisfied only if $B_1(x') = 0$ and $B_2(x') = 0$, i.e.,

$$(f_{2,x'}^2)_{x'} + 2(a_{x'}/a - f_{4,x'}/f_4)f_{2,x'}^2 = 2(a_{x'}a_{x'x'}/a^4 - a_{x'}^2 f_{4,x'}/a^4 f_4 - a_{x'}^3/a^5)\ , \qquad \text{(A22)}$$

$$-a_{x'}f_{4,x'}/a + f_4(a_{x'x'}/a - a_{x'}^2/a^2 + a^2 f_{2,x'}^2) = 0\ , \qquad \text{(A23)}$$

where subscripts x' denote derivatives with respect to x'.

Equation (A22) can be solved for $f_{2,x'}{}^2$:

$$f_{2,x'}^2 = a_{x'}^2/a^4 - kf_4^2/a^2\ , \qquad \text{(A24)}$$

where k is an integration constant.

Substituting (A24) into (A23) provides a Bernoulli equation for f_4,

$$-a_{x'}f_{4,x'} + a_{x'x'}f_4 - kaf_4^3 = 0$$

which can be solved to

$$f_4 = \epsilon a_{x'}(h + ka^2)^{-1/2}\ , \qquad \text{(A25)}$$

where h is another integration constant and ϵ is the sign of $a_{x'}$ (because f_4 must be positive).

This expression of f_4, substituted into Eq. (A24), leads to Eq. (A8).

APPENDIX 3. DERIVATION OF EQS. (A18) AND (A19)

The distribution of accelerations may be described by

$$a(x') = a(0)\alpha(x')\ , \qquad \text{(A26)}$$

where $\alpha(x')$ is a dimensionless function of x' such that $\alpha(0) = 1$.

The limiting form of Eqs. (A13) must be $x = x'$, $t = t'$ when the acceleration vanishes everywhere, i.e., when $a(0)$ tends to 0.

This condition, applied to (A13b), can only be satisfied if one of the constants $h^{1/2}$ and k tends to zero as $a^2(0)$, and the other one at least as $a^2(0)$. Therefore, let be

$$h^{1/2} = \lambda a^2(0)\ ; \quad k = \mu a^2(0)\ . \qquad \text{(A27)}$$

It is easy to see that λ and μ are dimensionless constants. Since $a(0)$ and c are the only constants available in the present problem, λ and μ must be pure numbers.

When the expressions (A26) and (A27) are carried into Eqs. (A13), the limiting procedure, applied to t, yields

$$2\lambda + \mu = 1\ . \qquad \text{(A28)}$$

Since the following condition must be required to hold,

$$\lim_{a(0)\to 0} f_4 = 1\ ,$$

Eq. (A7) is such that, when $a(0)$ tends to 0, the acceleration function $\alpha(x')$ must behave as

$$1 + \epsilon a(0)x'(1 - \lambda)\ . \qquad \text{(A29)}$$

It can then be seen that if the conditions (A28) and (A29) are satisfied, x actually tends to x' when $a(0)$ tends to 0.

APPENDIX 4. PROOF OF THE LAST STATEMENT IN SEC. VII.2

As can be inferred from Eqs. (5.11), (7.2), and (7.4), the question is to decide whether D^2/μ^2 can be put into the form

$$H(x',t') = G(t'-x')G(t'+x')/G(t-x)G(t+x)$$

with a nonvanishing μ. If the partial derivative of H with respect to x' is computed, and then x' and t' are set equal to zero, the result is, when (A15) is used,

$$H_{x'}(0,0) = -2[G'(0)/G(0)]t_{x'}(0,0) = 0\,, \quad \text{(A30)}$$

where G' is the derivative of G with respect to its argument. When due account is taken of (7.2), it is easily seen that the condition (A30), applied to $H = D^2/\mu^2$, cannot be satisfied unless μ vanishes.

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Fourier Analysis of X-Ray Diffraction Data from Liquids

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CONTENTS

1. INTRODUCTION

THAT structural information about the liquid state can be obtained by application of x-ray diffraction techniques, has been known since Debye (1915) and Ehrenfest (1915) showed that the periodicity of a crystal structure is not required for the production of diffraction effects. Early experimental diffraction work was done by Debye and Scherrer (1916) on benzene and by Keesom and de Smedt (1923) on liquid argon. This was followed by the introduction by Debye (1927) of the concept of a probability function for the distribution of intermolecular distances. The relation of this function to the production of the diffraction pattern was discussed by Zernike and Prins (1927). These authors also showed how to apply the Fourier integral theorem to the determination of the probability function from diffraction patterns. Debye and Menke (1930, 1931) made the first quantitative application when they treated the case of liquid mercury.

X-ray diffraction measurements were made in a large number of liquids during the first half of this century. In a few instances experimental work was done over a range of pressure and temperature. The results for liquids are summarized in reviews by Gingrich (1943), Furukawa (1962), and Kruh (1962).

The techniques of data treatment by a number of authors were summarized in a book by Randall (1934). The method most commonly used at the present time is that of Warren and Gingrich (1934). A more general approach which treats subtle mathematical points with considerably more elegance is that of Filipovich (1955a, 1955b, 1956a, 1956b). This author rigorously presented the diffraction formulas in terms of both the radial atomic density and the radial electron density. He related these two functions and quantitatively treated the "diffraction error" caused by truncating the formal infinite integral required for the Fourier transformation of the intensity data. Filipovich also wrote expressions for the effect of improper normalization of experimental data. In a treatment applicable to the truncation error as a special case, Waser and Schomaker (1953) discussed the use of intensity data weighting

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